MOISTURE TRANSFER OF CERTAIN BODIES WITH VOLUMETRIC MOISTURE ABSORPTION

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An analytical solution is given to a boundary problem of the third kind, that of moisture transfer for bodies in the form of an infinite plate and an infinite cylinder, based on the generalized equation of moisture conduction, suggested by A. V. Luikov [1].

For a wall and for the cylinder the problem is formulated as follows:

differential equation

$$\tau_{rm}\frac{\partial^2 u}{\partial\tau^2}+\frac{\partial u}{\partial\tau}=a_m\left(\frac{\partial^2 u}{\partial\xi^2}+\frac{i}{\xi}\frac{\partial u}{\partial\xi}\right)-\frac{1}{\gamma_0}q(\tau)$$
$$(i=0,\ 1); \quad (1)$$

initial conditions

$$u(\xi,\ 0)=\psi_1(\xi),\quad \frac{\partial u(\xi,\ 0)}{\partial\tau}=\psi_2(\xi). \quad (2)$$

boundary conditions

$$\frac{\partial u(R_j,\ \tau)}{\partial\xi}=(-1)^j h_j\left[\varphi_j(\tau)-u(R_j,\ \tau)\right]\quad (j=1,\ 2). \quad (3)$$

We obtain a solution of problem (1)–(3) with the method of finite integral transformations due to G. A. Grinberg [2], having determined the integral transformations by the formulas

$$\bar u=\int_{R_1}^{R_2}u\,\xi^i W\left(\mu_n\frac{\xi}{L}\right)d\xi, \quad (4)$$

$$u=\sum_{n=1}^{\infty}\frac{\bar u W\left(\mu_n\frac{\xi}{L}\right)}{\int_{R_1}^{R_2}\xi^i W^2\left(\mu_n\frac{\xi}{L}\right)d\xi}. \quad (5)$$

The result is

$$u=\sum_{n=1}^{\infty}\frac{1}{B_n}W\left(\mu_n\frac{\xi}{L}\right)\Big\{\exp(-\tau/2\tau_{rm})\Big[\bar\psi_1\operatorname{ch}\frac{\lambda}{2}\tau+$$
$$+\frac{1}{\lambda\tau_{rm}}(\bar\psi_1+2\tau_{rm}\bar\psi_2)\operatorname{sh}\frac{\lambda}{2}\tau\Big]+$$
$$+\frac{2}{\lambda}\int_0^{\tau}\Phi(\zeta)\exp\left[(\zeta-\tau)/2\tau_{rm}\right]\operatorname{sh}\frac{\lambda}{2}(\tau-\zeta)\,d\zeta\Big\}, \quad (6)$$

where

$$\Phi(\tau)=\frac{a_m}{\tau_{rm}}\Big[R_2^i h_2\varphi_2(\tau)W\left(\mu_n\frac{R_2}{L}\right)+$$
$$+R_1^i h_1\varphi_1(\tau)W\left(\mu_n\frac{R_1}{L}\right)-\frac{1}{a_m\gamma_0}q(\tau)N_n\Big], \quad (7)$$

$$N_n=\int_{R_1}^{R_2}\xi^i W\left(\mu_n\frac{\xi}{L}\right)d\xi, \quad (8)$$

$$B_n=\int_{R_1}^{R_2}\xi^i W^2\left(\mu_n\frac{\xi}{L}\right)d\xi. \quad (9)$$

In calculations in accordance with (6), the sum $\sum_{n=1}^{\infty}$ is divided into two subsums,

$$\sum_{n=1}^{\infty}=\sum_{n=1}^{s}+\sum_{n=s+1}^{\infty}. \quad (10)$$

The number s is chosen such that

$$\mu_s^2<\frac{L^2}{4a_m\tau_{rm}}<\mu_{s+1}^2. \quad (11)$$

In the first subsum we put

$$\lambda=\lambda_1=\frac{1}{\tau_{rm}}\sqrt{1-\frac{4\mu_n^2 a_m\tau_{rm}}{L^2}}, \quad (12)$$

and in the second

$$\lambda=i\lambda_1\ (\text{here } i=\sqrt{-1}). \quad (13)$$

In the rare event that

$$\mu_s^2=\frac{L^2}{4a_m\tau_{rm}}, \quad (14)$$

the last term of the first subsum should be replaced by the term

$$\frac{1}{B_n}W\left(\mu_s\frac{\xi}{L}\right)\Big\{\exp(-\tau/2\tau_{rm})\Big[\bar\psi_1+\frac{1}{2\tau_{rm}}(\bar\psi_1+$$
$$+2\tau_{rm}\bar\psi_2)\Big]+\int_0^{\tau}\Phi(\zeta)(\zeta-\tau)\exp\left[(\zeta-\tau)/2\tau_{rm}\right]d\zeta\Big\}. \quad (15)$$

Let

$$\varphi_j(\tau)=u_j=\text{const}\ (j=1,\ 2);\ \psi_1=u_0=\text{const};$$
$$\psi_2=0;\ q(\tau)=q_0\exp(-\alpha\tau).$$

An exponential dependence of the intensity of volume moisture absorption on time occurs, for example, in the hydration of concrete. Then

$$u=u_2+(u_2-u_1)v(\xi)-\frac{q_0}{a_m\gamma_0}w(\xi)-$$
$$-L\exp(-\tau/2\tau_{rm})\sum_{n=1}^{\infty}\frac{1}{\mu_n^2 B_n}W\left(\mu_n\frac{\xi}{L}\right)\times$$

$$\times\left[R_2^i\,\mathrm{Bi}_2(u_2-u_0)\,W\left(\mu_n\frac{R_2}{L}\right)+\right.$$

$$\left.+R_1^i\,\mathrm{Bi}_1(u_1-u_0)\,W\left(\mu_n\frac{R_1}{L}\right)\right]\times$$

$$\times\left(\mathrm{ch}\,\frac{\lambda}{2}\,\tau+\frac{1}{\lambda\tau_{rm}}\,\mathrm{sh}\,\frac{\lambda}{2}\,\tau\right)+\frac{q_0L^2}{a_m\gamma_0}\times$$

$$\times\exp(-\tau/2\tau_{rm})\sum_{n=1}^{\infty}\frac{W\left(\mu_n\frac{\xi}{L}\right)N_n}{B_n\left[\mu_n^2-\frac{\alpha L^2}{a_m}(1-\alpha\tau_{rm})\right]}\times$$

$$\times\left(\mathrm{ch}\,\frac{\lambda}{2}\,\tau+\frac{1-2\alpha\tau_{rm}}{\lambda\tau_{rm}}\,\mathrm{sh}\,\frac{\lambda}{2}\,\tau\right). \qquad (16)$$

Here

$$v(\xi)=\sum_{n=1}^{\infty}\frac{L}{\mu_n^2B_n}\,W\left(\mu_n\frac{\xi}{L}\right)\left[R_2^i\,\mathrm{Bi}_2(u_2-u_0)\,W\times\right.$$

$$\left.\times\left(\mu_n\frac{R_2}{L}\right)+R_1^i\,\mathrm{Bi}_1(u_1-u_0)\,W\left(\mu_n\frac{R_1}{L}\right)\right]; \qquad (17)$$

$$w(\xi)=\sum_{n=1}^{\infty}\frac{L^2N_nW\left(\mu_n\frac{\xi}{L}\right)}{B_n\left[\mu_n^2-\frac{\alpha L^2}{a_m}(1-\alpha\tau_{rm})\right]}. \qquad (18)$$

It is not difficult to show that the sums of series (17) and (18) are determined as solutions of the following problems:

$$\left.\begin{aligned}&\frac{d^2v}{d\xi^2}+\frac{i}{\xi}\frac{dv}{d\xi}=0\quad(i=0,\ 1),\\&\frac{dv(R_j)}{d\xi}=(-1)^{j+1}h_j\left[(2-j)+v(R_j)\right]\quad(j=1,\ 2)\end{aligned}\right\} \qquad (19)$$

and

$$\left.\begin{aligned}&\frac{d^2w}{d\xi^2}+\frac{i}{\xi}\frac{dw}{d\xi}+\frac{\alpha}{a_m}(1-\alpha\tau_{rm})\,w=-1\\&\qquad(i=0,\ 1),\\&\frac{dw(R_j)}{d\xi}=(-1)^{j+1}h_jw(R_j)\quad(j=1,\ 2).\end{aligned}\right\} \qquad (20)$$

In a number of cases, the solutions of problems (19) and (20) are cumbersome, and evaluation of $v(\xi)$ and $w(\xi)$ from formulas (17) and (18) in the general context of the solution of (16) is simpler. We shall now specify the solutions obtained.

1. Plate with asymmetric moisture transfer ($i = 0$, $\xi = x$, $R_1 = 0$, $R_2 = R = L$ is the plate width).

The eigenfunction of the problem is

$$W\left(\mu_n\frac{\xi}{L}\right)=X_n=\mu_n\cos\mu_n\frac{x}{R}+\mathrm{Bi}_1\sin\mu_n\frac{x}{R}. \qquad (21)$$

The characteristic equation is

$$\mathrm{ctg}\,\mu_n=\frac{\mu_n^2-\mathrm{Bi}_1\mathrm{Bi}_2}{\mu_n(\mathrm{Bi}_1+\mathrm{Bi}_2)}. \qquad (22)$$

The integrals are

$$N_n=\frac{R}{\mu_n}\left[(-1)^{n+1}\mathrm{Bi}_2\sqrt{\frac{\mu_n^2+\mathrm{Bi}_1^2}{\mu_n^2+\mathrm{Bi}_2^2}}+\mathrm{Bi}_1\right], \qquad (23)$$

$$B_n=\frac{R}{2}\left[\mu_n^2+\mathrm{Bi}_1^2+\mathrm{Bi}_1+\frac{\mathrm{Bi}_2(\mu_n^2+\mathrm{Bi}_1^2)}{\mu_n^2+\mathrm{Bi}_2^2}\right]. \qquad (24)$$

The functions are

$$v(x)=\frac{\mathrm{Bi}_1}{(\mathrm{Bi}_1\mathrm{Bi}_2+\mathrm{Bi}_1+\mathrm{Bi}_2)}\left(\mathrm{Bi}_2\frac{x}{R}-\mathrm{Bi}_2-1\right) \qquad (25)$$

$$w(x)=\frac{R^2}{m^2}\left\{\frac{\left[\mathrm{Bi}_2-X_1(m,\ \mathrm{Bi}_2)\right]}{\left[\mathrm{Bi}_1X(m,\mathrm{Bi}_2)+mX_1(m,\mathrm{Bi}_2)\right]}\times\right.$$

$$\left.\times X\left(m\frac{x}{R},\ \mathrm{Bi}_1\right)+\cos m\frac{x}{R}-1\right\}. \qquad (26)$$

In formula (26) the following notation is introduced:

$$X\left(m\frac{x}{R},\ \mathrm{Bi}_j\right)=m\cos m\frac{x}{R}+\mathrm{Bi}_j\sin m\frac{x}{R}, \qquad (27)$$

$$X_1\left(m\frac{x}{R},\ \mathrm{Bi}_j\right)=-m\sin m\frac{x}{R}+\mathrm{Bi}_j\cos m\frac{x}{R}, \qquad (28)$$

$$m=\frac{\alpha L^2}{a_m}(1-\alpha\tau_{rm}). \qquad (29)$$

2. Plate with symmetrical moisture transfer ($i = 0$, $\xi = x$, $R_1 = 0$, $h_1 = 0$, $R_2 = R = L$ is the plate half-width). The solution is the same as in Part 1 with $\mathrm{Bi}_1 = 0$.

3. Hollow cylinder ($i = 1$, $\xi = r$, R_1 is the radius of the inner surface, R_2 is the radius of the outer surface, and $L = R_1$).

The eigenfunction of the problem is

$$U_0\left(\mu_n\frac{r}{R_1}\right)=\left[Y_0(\mu_n)+\frac{\mu_n}{\mathrm{Bi}_1}Y_1(\mu_n)\right]I_0\left(\mu_n\frac{r}{R_1}\right)-$$

$$-\left[I_0(\mu_n)+\frac{\mu_n}{\mathrm{Bi}_1}I_1(\mu_n)\right]Y_0\left(\mu_n\frac{r}{R_1}\right). \qquad (30)$$

The characteristic equation is

$$\frac{U_0(k\mu_n)}{U_1(k\mu_n)}=\frac{\mu_n}{\mathrm{Bi}_2}, \qquad (31)$$

where

$$U_\nu(k\mu_n)=\left[Y_0(\mu_n)+\frac{\mu_n}{\mathrm{Bi}_1}Y_1(\mu_n)\right]I_\nu(k\mu_n)-$$

$$-\left[I_0(\mu_n)+\frac{\mu_n}{\mathrm{Bi}_1}I_1(\mu_n)\right]Y_\nu(k\mu_n)\quad(\nu=0,\ 1) \qquad (32)$$

The integrals are

$$N_n=\frac{R_1^2}{\mu_n^2}\left[k\mathrm{Bi}_2U_0(k\mu_n)-\frac{2}{\pi}\right], \qquad (33)$$

$$B_n=\frac{R_1^2}{2\mu_n^2}\left[k^2U_0^2(k\mu_n)(\mu_n^2+\mathrm{Bi}_2^2)-\right.$$

$$\left.-\frac{4}{\pi^2\mathrm{Bi}_1^2}(\mu_n^2+\mathrm{Bi}_1^2)\right]. \qquad (34)$$

The functions are

$$v(r) = \frac{k\,\mathrm{Bi}_2}{(k\,\mathrm{Bi}_2 + \mathrm{Bi}_1 + k\,\mathrm{Bi}_1\mathrm{Bi}_2 \ln k)}\left(\mathrm{Bi}_1 \ln \frac{r}{R_1} + 1\right), \quad (35)$$

$$w(r) = \frac{R_1^2}{m^2}\left\{\frac{U_0\left(m\frac{r}{R_1}\right)}{\left[U_0(km) - \frac{m}{\mathrm{Bi}_2}U_1(km)\right]} - \right.$$

$$-\left\{\left[Y_0(km) - \frac{m}{\mathrm{Bi}_2}Y_1(km)\right]I_0\left(m\frac{r}{R_1}\right) + \right.$$

$$\left. + \left[I_0(km) - \frac{m}{\mathrm{Bi}_2}Y_1(km)\right]Y_0\left(m\frac{r}{R_1}\right)\right\} \times$$

$$\left. \times \left[U_0(km) - \frac{m}{\mathrm{Bi}_2}U_1(km)\right]^{-1} - 1\right\}. \quad (36)$$

4. Solid cylinder ($i = 1$, $\xi = r$, $R_1 = 0$, $h_1 = 0$, $R_2 = R = L$).

The eigenfunction of the problem is

$$W\left(\mu_n \frac{\xi}{L}\right) = I_0\left(\mu_n \frac{r}{R}\right). \quad (37)$$

The characteristic equation is

$$\frac{I_0(\mu_n)}{I_1(\mu_n)} = \frac{\mu_n}{\mathrm{Bi}}. \quad (38)$$

The integrals are

$$N_n = \frac{R^2}{\mu_n} I_1(\mu_n), \quad (39)$$

$$B_n = \frac{R^2 I_0^2(\mu_n)}{2\mu_n^2}(\mu_n^2 + \mathrm{Bi}^2). \quad (40)$$

The functions are

$$v(r) = 0, \quad (41)$$

$$w(r) = \frac{R^2}{m^2}\left[\frac{I_0\left(m\frac{r}{R}\right)}{I_0(m) - \frac{m}{\mathrm{Bi}}I_1(m)} - 1\right]. \quad (42)$$

NOTATION

u is the moisture content; τ is time; ξ is a coordinate; a_m is the coefficient of moisture diffusion in the body; γ_0 is the density of the perfectly dry body; q is the intensity of volumetric moisture absorption; τ_{rm} is the period of moisture transfer relaxation; h_j ($j = 1, 2$) is the relative moisture transfer coefficient; R_j are the coordinates of the finite body surfaces; $k = R_2/R_1$; L is the characteristic body dimension; $\mathrm{Bi}_1 = h_1 L$, $\mathrm{Bi}_2 = h_2 L$; $W(\mu_n \xi/L)$ is the eigenfunction of the problem; μ_n is the root of the characteristic equation.

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